The Pythagorean Theorem

Fermat's Last Theorem is one of the most famous theorems in the history of mathematics. It states that:

> *It is impossible to separate any power higher than the second into two like powers, or, using more formal mathematical notation: If an integer n is greater than 2, then* $a^n + b^n = c^n$ *has no solutions in non-zero integers a, b, and c.*

Despite how closely the problem is related to the Pythagorean theorem, which has infinite solutions and hundreds of proofs, Fermat's subtle variation is much more difficult to prove. The 17th-century mathematician Pierre de Fermat wrote in 1637 in his copy of Claude-Gaspar Bachet's translation of the famous *Arithmetica* of Diophantus: "I have a truly marvelous proof of this proposition which this margin is too narrow to contain." However, no correct proof was found for 357 years, until it was finally proven using very deep methods by Andrew Wiles in 1995 (after a failed attempt a year before). All the other theorems proposed by Fermat were eventually proven or disproven, either in his own proofs or by other mathematicians, in the two centuries following their proposition. The theorem was *not* the last that Fermat conjectured, but the *last to be proven*.

As a result of Fermat's marginal note, the proposition that the Diophantine equation $x^n + y^n = z^n$ where x, y, and z are rational numbers, and n is an integer, has no nonzero solutions for n > 2 has come to be known as **Fermat's Last Theorem**. It was called a "theorem" on the strength of Fermat's statement, despite the fact that no other mathematician was able to prove it for hundreds of years.

Fermat's original note in Latin

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

English translation

It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.

Some Selected Pythagorean Triples

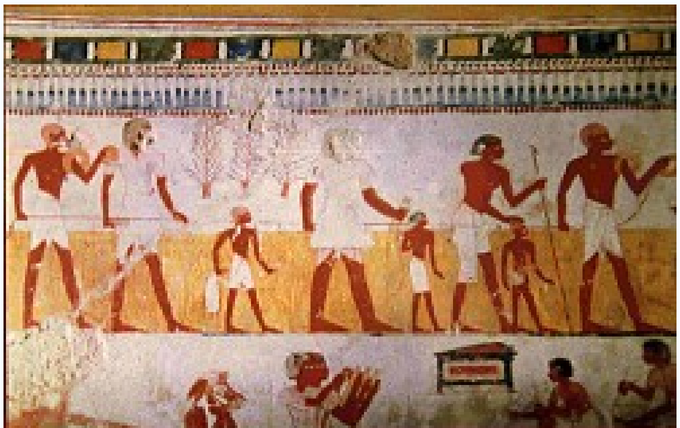
a	b	c		a	b	c		a	b	c
3	4	5		6	8	10		20	21	29
5	12	13		8	15	17		28	45	53
7	24	25		10	24	26		33	56	65
9	40	41		12	35	37		36	77	85
11	60	61		14	48	50		39	80	89
13	84	85		16	127	129		48	55	73
15	112	113		18	80	82		65	72	97
17	144	145		20	99	101		56	90	106
19	180	181		22	120	122		60	91	109
21	220	221		24	143	145		44	117	125
23	264	265		26	168	170		51	140	149
25	312	313		28	195	197		88	105	137
27	364	365		32	255	257		85	132	157
29	420	421		36	323	325		57	176	185
31	480	481		40	399	401		95	168	193
33	544	545		44	483	485		119	120	169

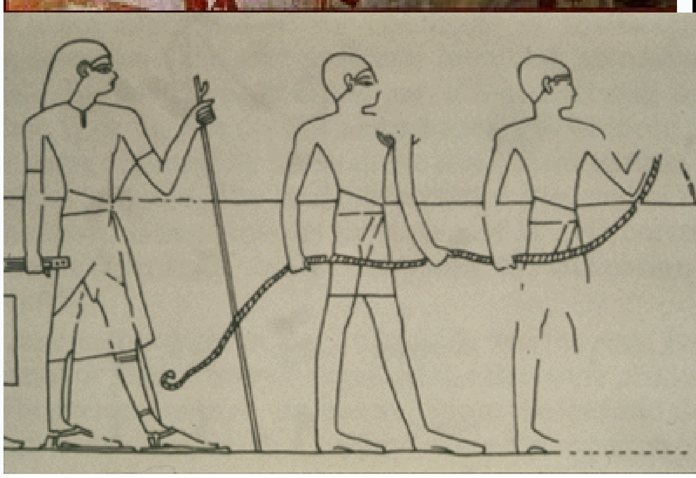
(all multiples of these triples also form Pythagorean triples)

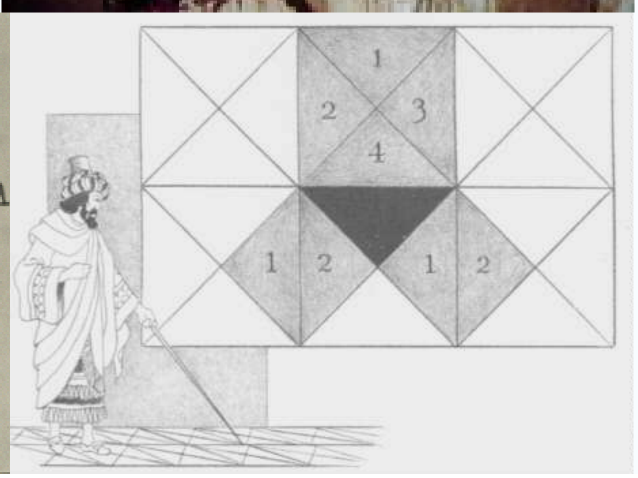
The Egyptian “Rope Stretchers” and Pytharoas

In ancient Egypt **Rope stretchers** were surveyors who measured property lines and foundations using knotted cords which they stretched in order to take the sag out of the rope. As far back as the palettes of Narmer and the Scorpion King the Egyptians document the process the royal surveyors used to restore the boundaries of fields after each innundation or flood.

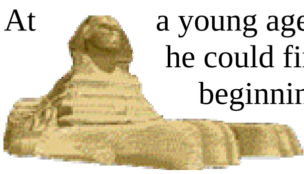
Rope stretchers used 3-4-5 triangles and the plumet, which are still in use by modern surveyors. The plummet can be used with a square ruled off into intervals on tongue and blade to get a unit rise and run or angle when taking an elevation to a distant point as with a modern sextant. Rope stretching technology spread to ancient Greece and India, where it stimulated the development of geometry and mathematics.





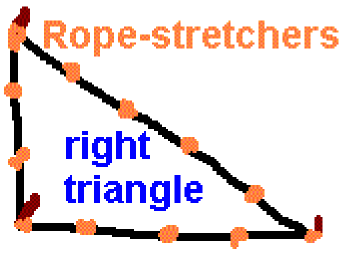


Pythagoras was born in the late 6th century B.C. on the island of Samos. His mother was most likely a Phoenician, his father a Greek stonecutter. He was one of those rare, truly genius types. He was very smart and thirsted for all the knowledge he could find.

At  a young age he left home to travel the known world and learn everything he could find. He studied under the Greek named Thales who was just beginning to discover the concepts of geometry. Thales encouraged the young Pythagoras to travel for himself in the ancient lands and study the development of learning at its source.

So Pythagoras went to Babylon and studied with the Chaldean stargazers. He went to Egypt and studied the lore of the priests at Memphis and Diospolis.

In Egypt he studied with the people known as the "rope-stretchers". These were the engineers who built the pyramids.

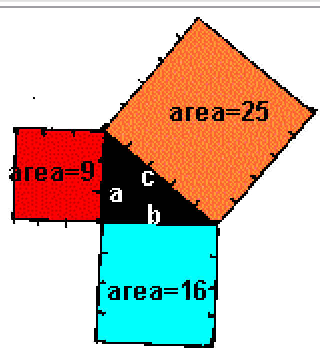


They held a very special secret in the form of a rope tied in a circle with 12 evenly spaced knots. It turns out that if the rope was pegged to the ground in the dimensions of 3-4-5, a right triangle would emerge instantly. This enabled them to lay the foundations for their buildings accurately. He traveled to all the known parts of the Mediterranean world. During his travels he came to the conclusion that the earth must be round. In history, he is given credit as the first person to spread this idea.

Pythagoras spent many years learning by travelling. Some say he made it the whole way to India and was deeply influenced, for he took up Oriental dress, including a turban. Many of his mystical ideas like number magic and reincarnation, were typical of the East. Finally he returned home. He was probably the single most educated man on the face of the earth at that point. He wanted to share what he knew, but the people of his home town Samos were less than enthusiastic. Tired of finding no one who would listen to his learning, he decided to "buy" a student. He found a homeless child and offered him a bribe. Pythagoras would pay him three obli for every lesson the boy mastered.

Now the boy thought this was great. He could sit all day in the shade of a large tree and listen to this old man and could make better wages than in a whole day's work in the hot sun. Naturally, he concentrated hard while Pythagoras introduced him to mathematical disciplines.
From the simple calculations of the Egyptian rope-stretchers, to the methods of the Phoenician navigators, to abstract rules and reasoning, Pythagoras led his pupil on. Soon the subject became so interesting that the boy begged for more and more lessons. At this point, Pythagoras explained that he could not afford to pay someone to just listen to him anymore. So they reached a bargain. The boy had saved enough to pay Pythagoras for his lessons. This was probably the start of organized education. Eventually Pythagoras left the island of Samos and settled on the Isle of Croton. This is where he formed his Secret Brotherhood. The Secret Brotherhood was a religious order with initiation rites and purifications and Pythagoras was its supreme unquestioned leader. He taught them that KWOWLEDGE WAS THE GREATEST PURIFICATION, and for them knowledge meant mathematics.

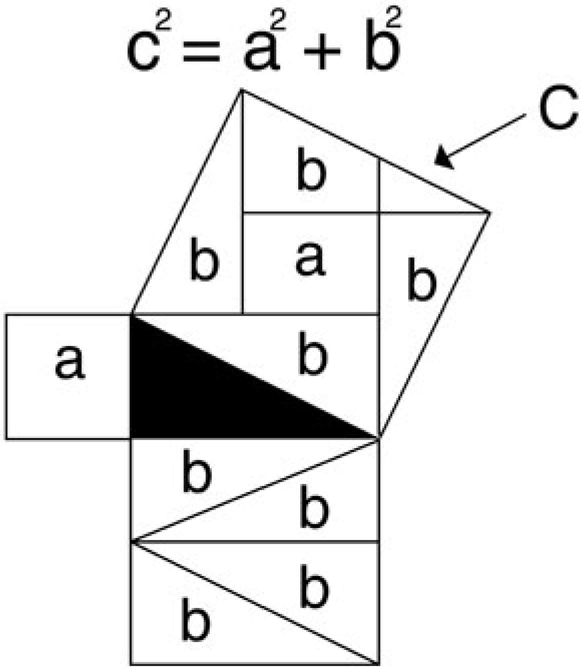
The most famous discovery that Pythagoras made came from his fascination with the Egyptian 3-4-5 rope-stretchers triangle. He had spent years thinking about it and what magic it might hold. Lo and behold, ... it DID hold a great deal of mathematics and for Pythagoras that was the same thing as magical power. One day while drawing in the sand he found that if a square is drawn from each side of the 3-4-5 triangle, the area of the two small squares added together equals the area of the large square.

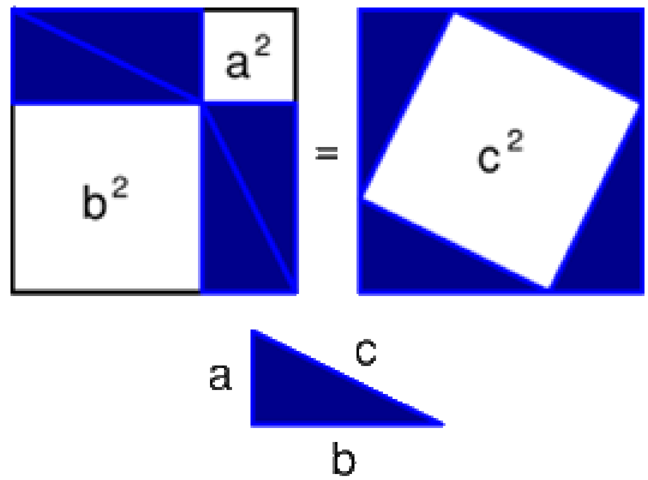


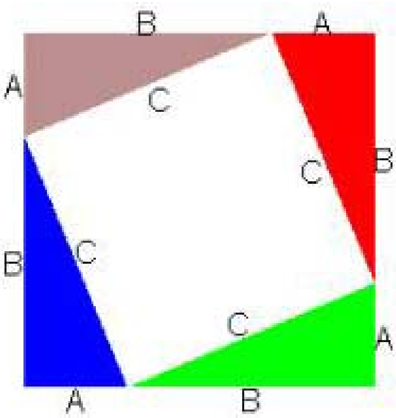
$3^2+4^2=5^2$ or $9 + 16 = 25$

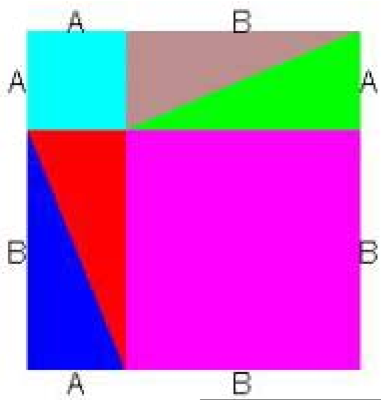
He examined other right triangles and found it was true with them also: $6^2 + 8^2 = 10^2$, $9^2 + 12^2 = 15^2$

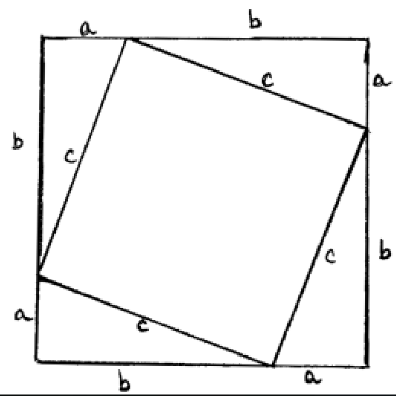
So he decided to announce it as a revelation from the god Apollo, who many claimed to be his father. When he revealed this finding to his followers, he used the general terms of ***a*** and ***b*** for the shorter legs and ***c*** for the longer side which he gave the name "**hypotenuse**". Thus we have the famous **Pythagorean Theorem!**

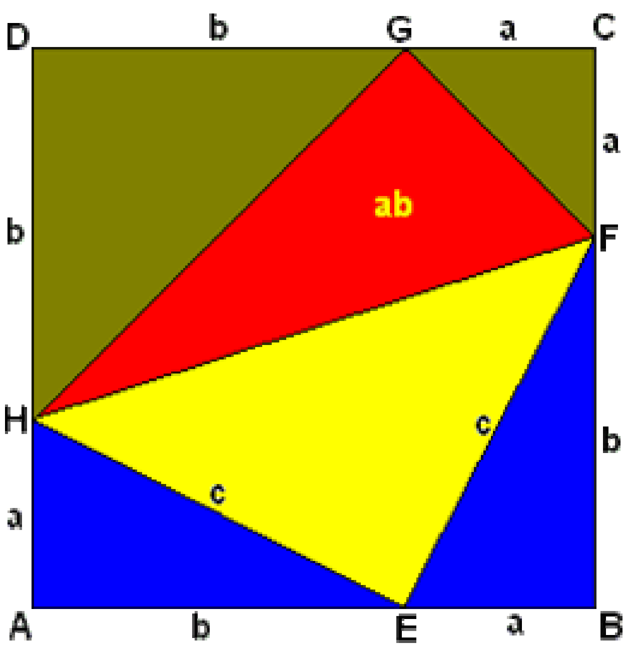

$c^2 = a^2 + b^2$
C
b
b
a
b
a
b
b
b
b
b
b

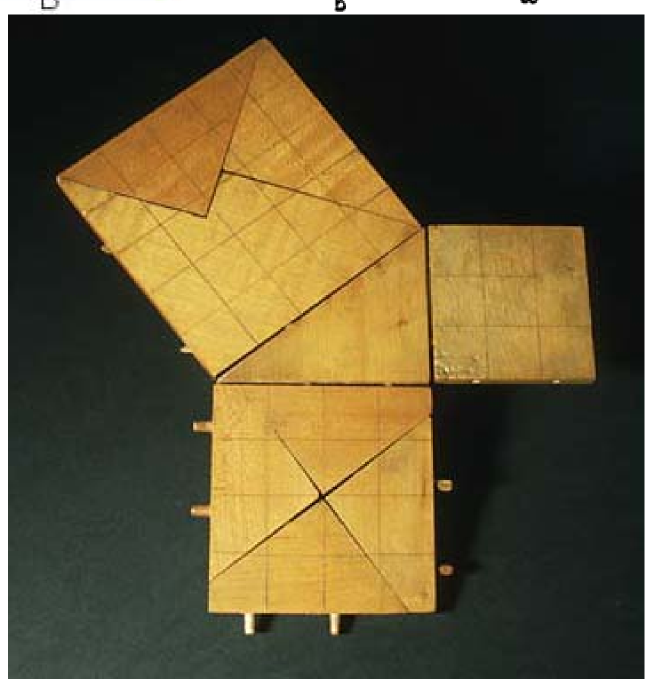

a^2
b^2
=
c^2
a
c
b

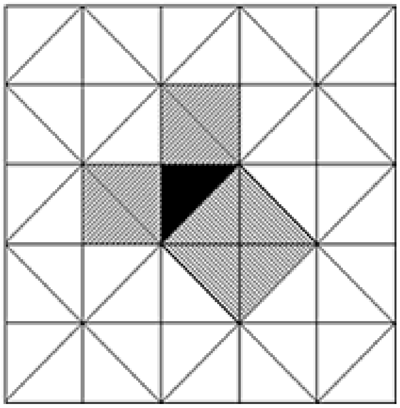

B A
A C
C B
B C
C A
A B

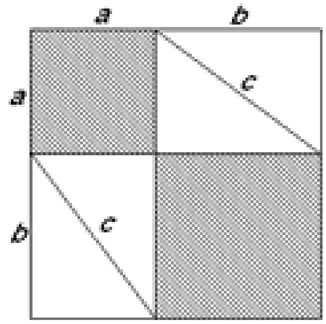

A B
A A
B B
A B


a b
a
b c
c
c b
a c
c
b a

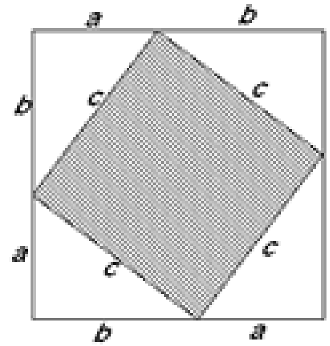

D b G a C
a
ab
b F
H
c b
c
a
A b E a B



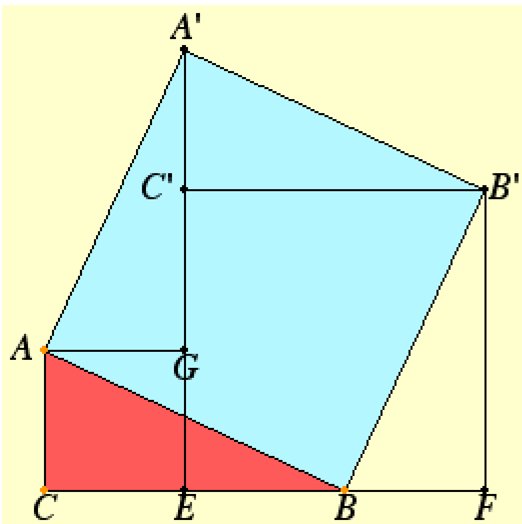




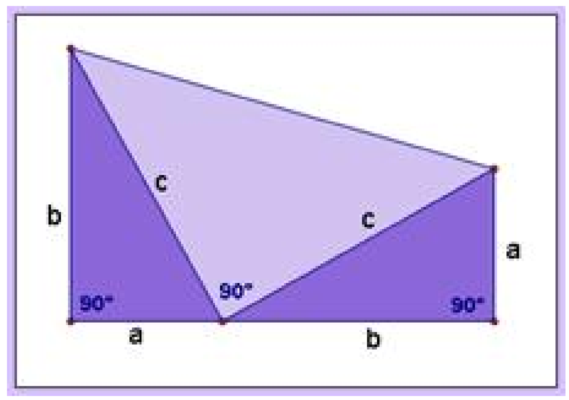
a
b
a
c
b
c



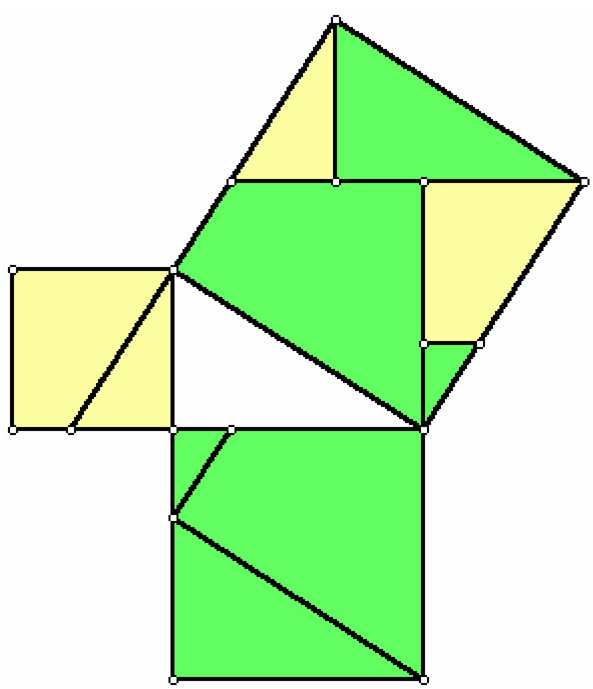
a
b
b
c
c
a
c
c
b
a

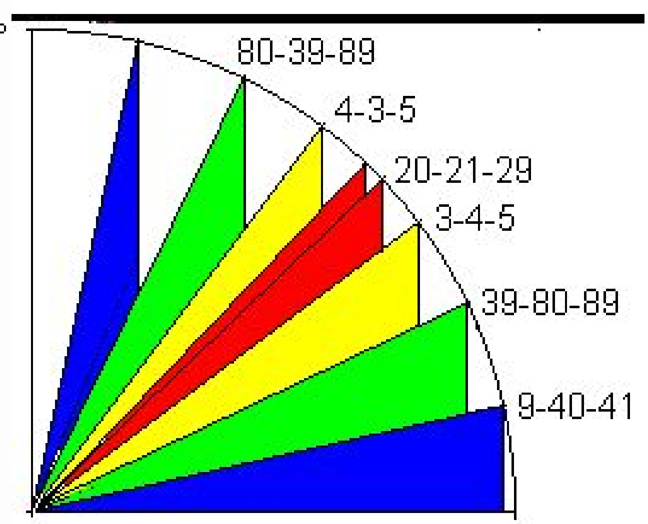


A'
C'
B'
A
G
C
E
B
F

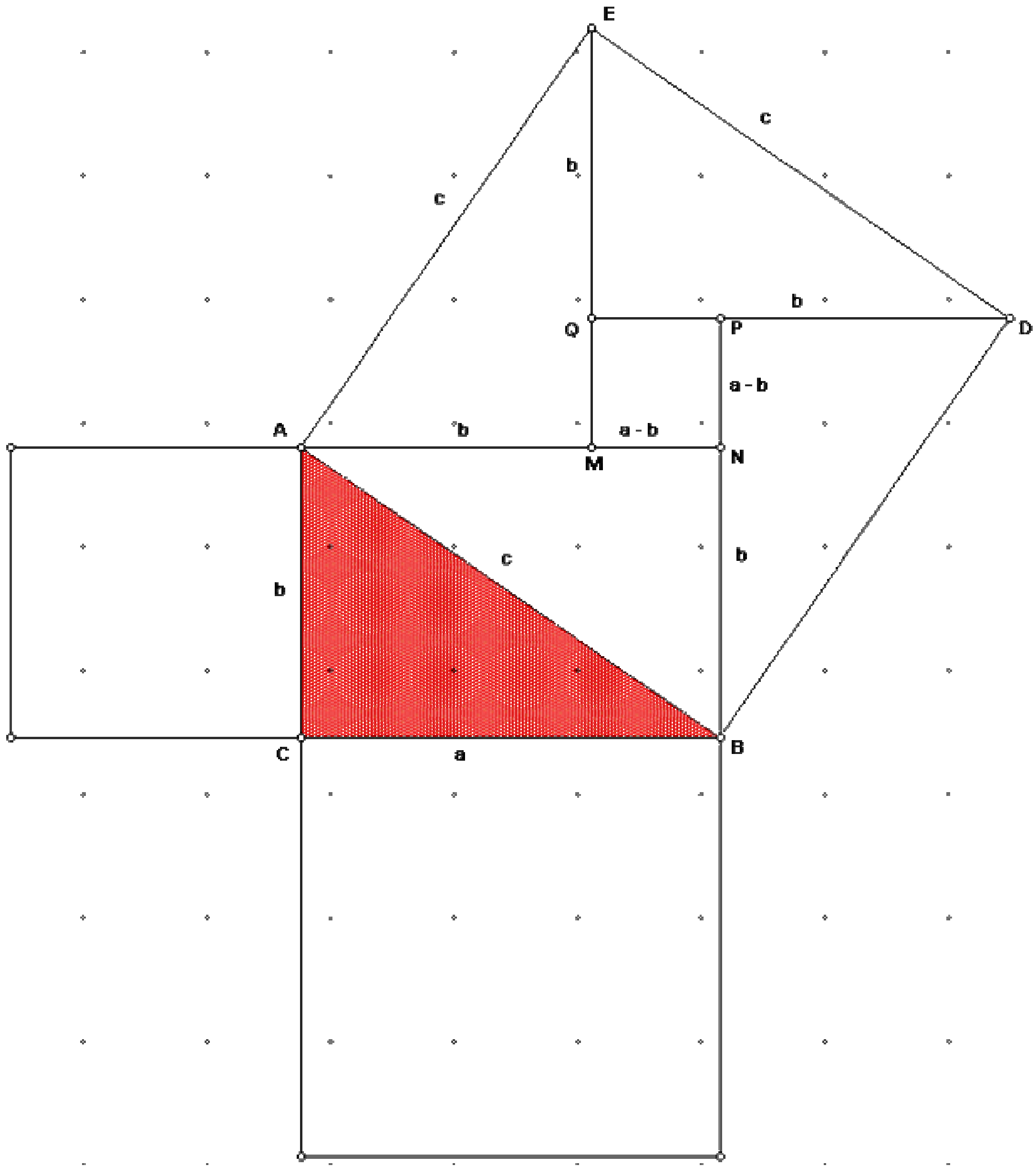


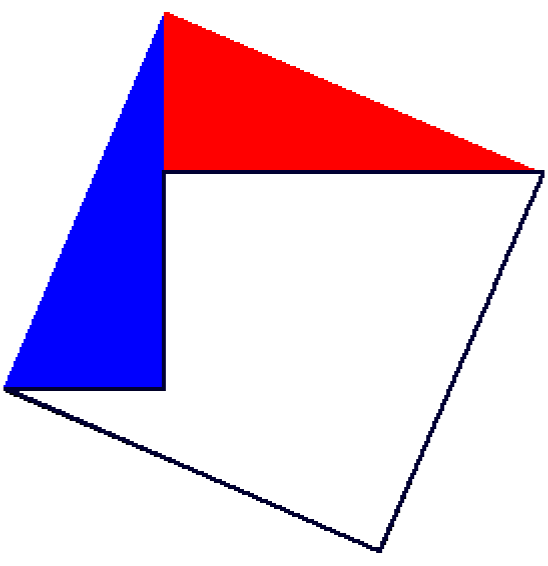
b
c
c
a
90°
90°
90°
a
b

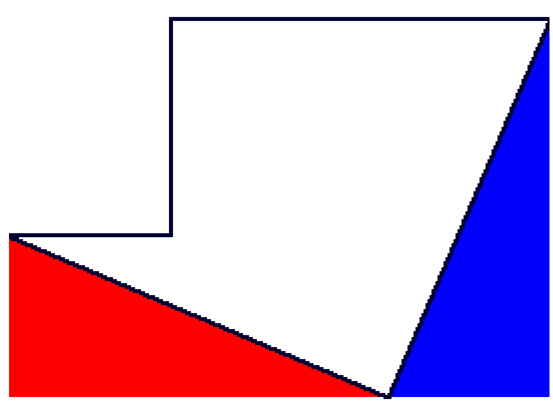


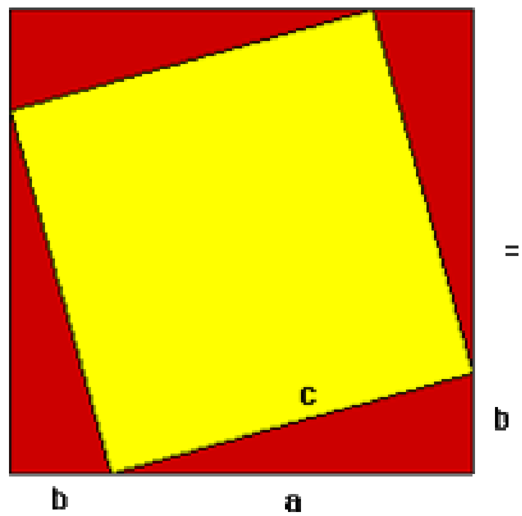


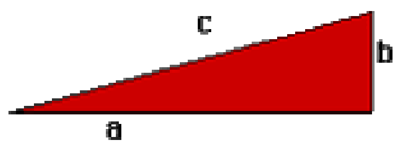
80-39-89
4-3-5
20-21-29
3-4-5
39-80-89
9-40-41

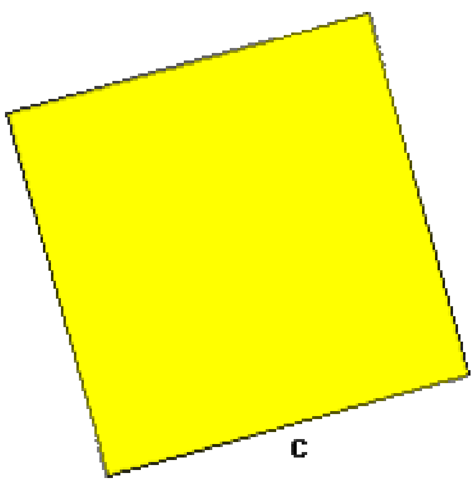

E
c
b
c
b
Q
P
D
a - b
A
b
a - b
M
N
c
b
b
C
a
B

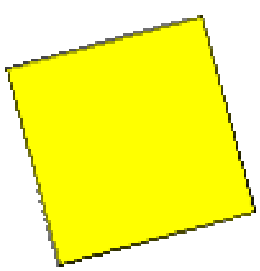
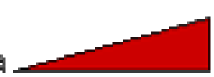
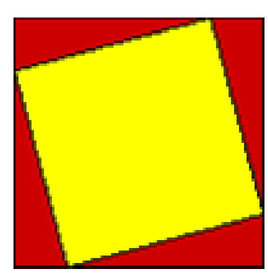




area of  = $(a+b)(a+b)$

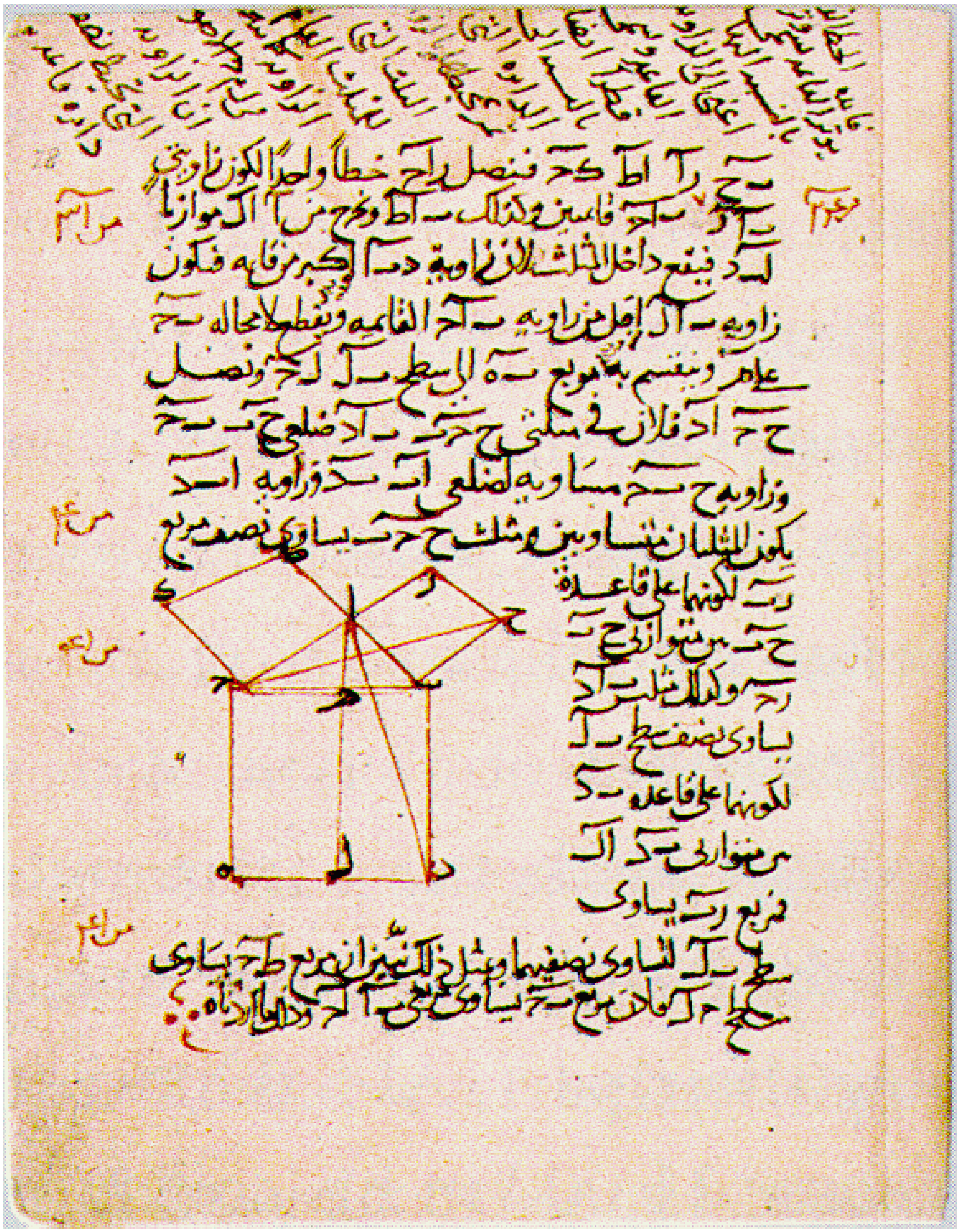
area of  = $ab/2$

area of  = c^2

(area ) + 4 (area ) = (area )

$$c^2 + 4\left(\frac{ab}{2}\right) = (a+b)^2$$

Nasrudeen Tusi's Record of Euclid's Proof of the Pythagorean Theorem



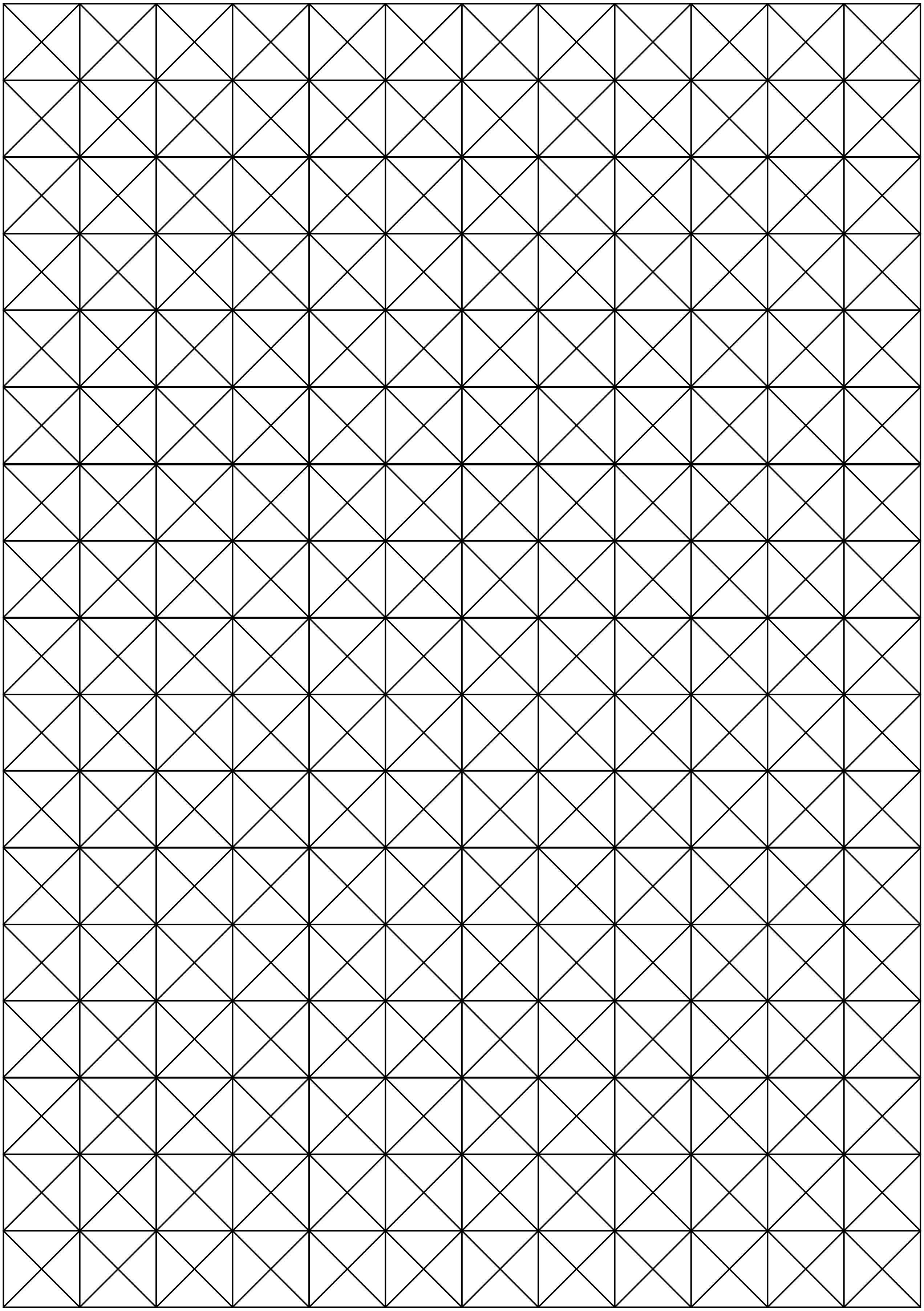
(see description on next page)

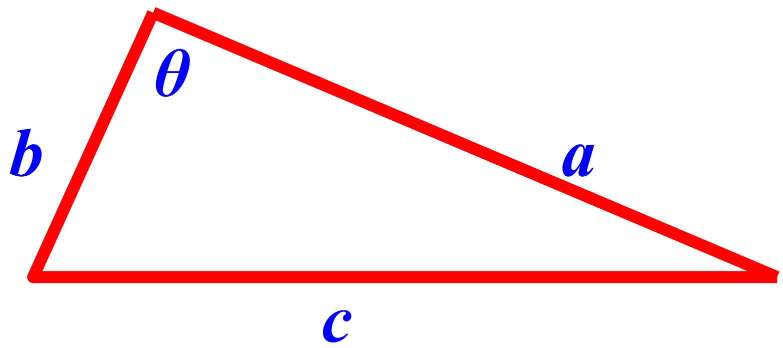
Nasr al-Din al-Tusi's Proof of the Pythagorean Theorem. Nasr al-Din al-Tusi (d. 1274 AD) was a renowned *Khorasani* Muslim Mathematician who reexamined Euclidean geometry [Khorasan is today's Iran/Afghanistan]. In this plate, one can read, in Arabic, Nasr al-Din al-Tusi's version of Euclid's proof of the Pythagorean Theorem. (see the discussion on p. 16 of [1] for a generalization and an analogy). It is claimed that the oldest proof goes back to the Chinese from about 3000 years ago (circa 1800 BC; what you will find in this hyperlinked slide is their version; it is the easiest proof). The oldest records of 'Pythagorean' numbers are found in clay tables dating back to the 1600-1800's BC found in Babylon, Iraq; see [2]. There are many proofs of this joyful fact online like here and here. In this document from about 900 years ago, we explain some of the features of "Arab" mathematics and offer a translation as well. Al-Tusi was not Arab. However, Arabic was the lingua franca of science in his time. This proof is already available in English online. It is said that Pythagoras (whose father was `Lebanese' and mother was `Greek', but spent most of his life and died as a `Sicilian' in Syracuse) learned his mathematics from the Babylonians. The statement of his theorem was found in their ancient texts.

This page from Tusi's work is very instructive about the development of the language of Mathematics. While Tusi uses *"Sat.h"* (roof or flat surface) to designate a rectangle, modern Arabic (and Farsi) uses *"Mustateel"* (elongated figure). *murabba*` [lit. quadre; square] was in fashion then and is used today as well. So is the case of a cube [ka`b as in the Kaabah in Mecca.]

The structure of the Mathematical language as we use it today can be traced in this plate.

The proof of a given statement is decomposed into a series of "maza`im" (claims) as indicated on both sides of the main text and argumentation based on known or previously established premises (see translation). One can see in the margin above the use of "rules" (Al-Hattani's *"Qaidah"*) and definitions (*"Musamma"*) and the ever-present *quod erat demonstrandum* Q.E.D. (*"wa thalika ma aradnah"*) announcing the end of the proof.





c	a	b	c^2	a^2	b^2	$a^2 + b^2$	$\angle\theta$
5	4	3					
6	5	4					
6	4	4					
9	7	6					
13	12	5					
15	10	9					
18	16	13					
17	15	8					
21	16	12					

c	a	b	c^2	a^2	b^2	$a^2 + b^2$	$\angle\theta$
5	4	3	25	16	9	25	90°
6	5	4	36	25	16	41	83°
6	4	4	36	16	16	32	97°
9	7	6	81	49	36	85	87°
13	12	5	169	144	25	169	90°
15	10	9	225	100	81	181	104°
18	16	13	324	256	169	425	76°
17	15	8	289	225	64	289	90°
21	16	12	441	256	144	400	96°

A very useful web site for various lists of numbers (Fibonacci, Pythagorean triples, perfect numbers, etc.) is: http://www.tsm-resources.com/alists/trip.html

From this page, the last few Pythagorean triples are shown here:

```
720     1961    2089
1254    1672    2090    [3 - 4 - 5]
459     2040    2091    [9 - 40 - 41]
555     2016    2091    [185 - 672 - 697]
984     1845    2091    [8 - 15 - 17]
1365    1584    2091    [455 - 528 - 697]
805     1932    2093    [5 - 12 - 13]
1080    1794    2094    [180 - 299 - 349]
1257    1676    2095    [3 - 4 - 5]
945     1872    2097    [105 - 208 - 233]
640     1998    2098    [320 - 999 - 1049]
588     2016    2100    [7 - 24 - 25]
1260    1680    2100    [3 - 4 - 5]
```

You will see that some are just magnifications of smaller ones where all the sides have been doubled, or trebled for example. The others are "new" and are usually called **primitive Pythagorean triples**. Any Pythagorean triangle is either primitive or a multiple of a primitive and this is shown in the table above.

Using the Fibonacci Numbers to make Pythagorean Triangles

There is an easy way to generate Pythagorean triangles using 4 Fibonacci numbers. Take, for example, the 4 Fibonacci numbers:

1, 2, 3, 5

Let's call the first two a and b. Since they are from the Fibonacci series, the next is the sum of the previous two: a+b and the following one is b+(a+b) or a+2b:-

a	b	a+b	a+2b
1	2	3	5

You can now make a Pythagorean triangle as follows:

1. Multiply the two middle or inner numbers (here 2 and 3 giving 6);
2. Double the result (here twice 6 gives **12**). This is one side, s, of the Pythagorean Triangle.
3. Multiply together the two outer numbers (here 1 and 5 giving **5**). This is the second side, t, of the Pythagorean triangle.
4. The third side, the longest, is found by adding together the *squares* of the inner two numbers (here 2^2=4 and 3^2=9 and their sum is 4+9=**13**). This is the third side, h, of the Pythagorean triangle.

We have generated the 12, 5,13 Pythagorean triangle, or, putting the sides in order, the **5, 12, 13** triangle this time.

Try it with 2, 3, 5 and 8 and check that you get the Pythagorean triangle: 30, 16, 34.
Is this one primitive?

In fact, this process works for **any two numbers a and b**, not just Fibonacci numbers. The third and fourth numbers are found using **the Fibonacci rule**: add the latest two values to get the next.

The sequence which generates the Pythagorean Triple Sequence (Type III):
(i.e., 0, 1, 2, 5, 12, 29, 70, 169, 408, . . .) can be expressed by the recursive formula:
$P_n = 2P_{n-1} + 1P_{n-2}$

In general, for the recursive sequence: T_n (for n from 1 to ∞), where $T_n = AT_{n-1} + BT_{n-2}$ (which is called a linear recurrence equation or formula), we have $T_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, where α and β are the roots of $x^2 = Ax + B$.

For the Fibonacci Sequence, F_n, $A = B = 1$, so $F_n = \dfrac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$

For the Pythagorean Triple (Type III) Sequence Generator, $A = 2$, $B = 1$, so we get:

$\boldsymbol{x}^2 = 2\boldsymbol{x} + 1$ and $x = \dfrac{2 \pm \sqrt{4-(-4)}}{2}$ or $x = \dfrac{2 \pm 2\sqrt{2}}{2}$ or $\boldsymbol{x} = 1 \pm \sqrt{2}$

Thus we get: $P_n = \dfrac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{1+\sqrt{2}-\left(1-\sqrt{2}\right)}$ or $P_n = \dfrac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$

(See Excel sheet for actual Pythagorean Triples (Type III) thus generated.)