

## Inverse Functions Attachment

It is essential to understand the difference between an inverse operation and an inverse function. In the simplest terms, inverse operations in mathematics are two operations that have the opposite effect, such as addition and subtraction or operations that undo each other. Addition and subtraction are inverse operations, as are multiplication and division. This can be confused with an inverse element (or inverse operation).

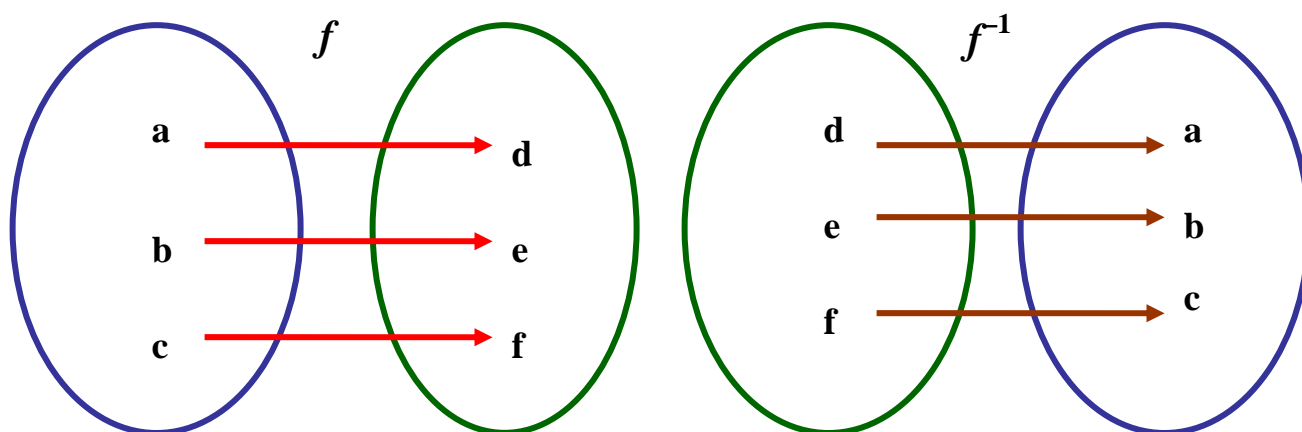
The **inverse element**, is defined for specific sets of numbers in relationship to a binary operation.. The formal definition of a group (of which one condition is that an inverse element exists) as it applies to set theory for binary operations follows:

A group  $(G, *)$  is a set  $G$  with a binary operation  $*$  that satisfies the following four axioms:

- *Closure* : For all  $a, b$  in  $G$ , the result of  $a * b$  is also in  $G$ .
- *Associativity*: For all  $a, b$  and  $c$  in  $G$ ,  $(a * b) * c = a * (b * c)$ .
- *Identity element*: There exists an element  $e$  in  $G$  such that for all  $a$  in  $G$ ,  $e * a = a * e = a$ .
- *Inverse element*: For each  $a$  in  $G$ , there exists an element  $b$  in  $G$  such that  $a * b = b * a = e$ , where  $e$  is an identity element.

For example, for the operation of addition for integers, the identity element is 0 (since we have  $a + 0 = 0 + a = a$  for all  $a \in \mathbb{Z}$ ) and the inverse of “ $a$ ” is “ $-a$ ” because  $a + (-a) = 0$ . This is also called the additive inverse. The multiplicative inverse (or the reciprocal) of “ $a$ ” is “ $\frac{1}{a}$ ”.

Using a mapping diagram, the inverse of  $f$  shown is  $f^{-1}$  (where  $-1$  is not an exponent).



Formally, if  $f$  is a function with domain  $X$ , then  $f^{-1}$  is its inverse function if and only if for every  $x \in X$  we have:  $f^{-1}[f(x)] = f[f^{-1}(x)] = x$ . We also get  $x = f^{-1}(y)$  for  $y = f(x)$ .

If a function  $f$  has an inverse then  $f$  is said to be **invertible**. If an inverse exists, it is unique. The superscript “ $-1$ ” is not an exponent.

We have these inverse functions in secondary school mathematics:

$$y = x + a \text{ and } y = x - a, y = ax \text{ and } y = \frac{x}{a}, y = x^2 \text{ and } y = \sqrt{x}, y = \sin x \text{ and } y = \sin^{-1}x,$$

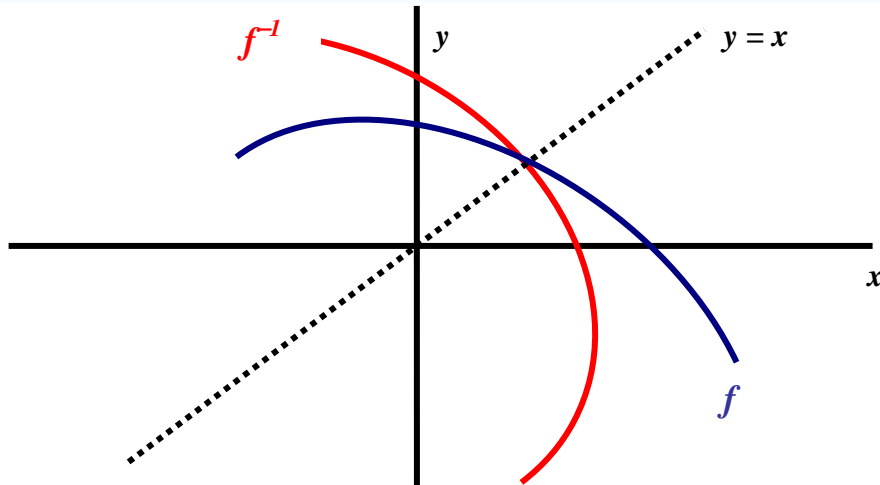
$$y = b^x \text{ and } y = \log_b x, y = e^x \text{ and } y = \ln x, y = f'(x) \text{ and } y = \int f(x) dx \text{ (no longer in our curriculum).}$$

An inverse function can be determined in three ways:

- i) using a set or list of elements, ii) using a graph, or iii) using an equation.

For a set of ordered pair elements, the inverse relation is obtained by interchanging the order of the elements in each pair. For example, if we have set  $A = \{(1, 5), (2, 7), (3, 11)\}$  then its inverse  $A^{-1}$  is  $\{(5, 1), (7, 2), (11, 3)\}$

The inverse of a relation defined by a graph is found by reflecting the graph in the line  $y = x$ . An example is shown below. In this case, the inverse is not a function. In general, the inverse of a function is a function if and only if the function is one-to-one. A one-to-one function is one whose graph passes both the vertical line test and the horizontal line test. Such a function is said to be **injective**. A function  $f$  is injective if it maps distinct  $x$  in the domain to distinct  $y$  in the codomain, such that  $f(x) = y$ . Put another way,  $f$  is injective if  $f(a) = f(b)$  implies  $a = b$  (or  $a \neq b$  implies  $f(a) \neq f(b)$ ), for any  $a, b$  in the domain.



The inverse of a relation given by a defining equation is found by interchanging the  $x$  and  $y$  variables and solving for  $y$ . The latter part of this process can often be challenging. If the intent is to have only an inverse function result, the domain must often be restricted in some fashion in order that the original function is one-to-one (as described above).

## Logarithms

The logarithm of a positive number  $N$  to a given base  $b$  (written  $\log_b N$ ) is the exponent of the power to which  $b$  must be raised to produce  $N$ . It is understood that  $b$  must be positive and not equal to 1. If base  $b$  is  $e$  (the base for natural logarithms) then it is written  $\ln N$  which is understood to be  $\log_e N$ . If the base is 10, it is called a common logarithm and  $\log N$  is understood to be  $\log_{10} N$ .

## Fundamental Laws for Logarithms

$$\log_b(PQR) = \log_b P + \log_b Q + \log_b R$$

$$\log_b(P^n) = n \log_b P$$

$$\log_b(\sqrt[n]{P}) = \frac{1}{n} \log_b P$$

$$\log_b(b) = 1$$

$$b^{\log_b x} = x$$

$$\log_b N = -\log_b \left( \frac{1}{N} \right)$$

$$\log_b \left( \frac{P}{Q} \right) = \log_b P - \log_b Q \quad (\text{provide that } Q \text{ is not zero})$$

$$\log_b P = \log_b Q \quad \text{only if } P = Q$$

$$\log_b(b^n) = n$$

$$\log_b(1) = 0$$

$$\log_b N = \frac{\log N}{\log b} = \frac{\ln N}{\ln b} \quad (\text{called the change of base formula})$$

$$\log_b N = \frac{1}{\log_N b}$$

and we change from exponential to logarithmic form using:  $\log_b x = y \Leftrightarrow x = by$

## Exponential Functions

The **exponential function** (one of the most important functions in mathematics) is written as  $y = e^x$ , where  $e$  equals approximately 2.71828183 and is the base of the natural logarithm. The exponential function is nearly flat (climbing slowly) for negative values of  $x$ , climbs quickly for positive values of  $x$ , and equals 1 when  $x$  is equal to 0. The feature which makes this function so useful in mathematics and a wide variety of applications in science technology, economics, geology, etc., is the fact that its  $y$  value always equals the slope at that point. Sometimes, the term **exponential function** is used for functions of the form  $y = ka^x$ , where  $a$ , called the *base*, is any positive real number.

As a function of the real variable  $x$ , the graph of  $y = e^x$  is always positive (above the  $x$  axis) and increasing (viewed left-to-right). It never touches the  $x$  axis, although it gets arbitrarily close to it (thus, the  $x$  axis is a horizontal asymptote to the graph). Its inverse function, the natural logarithm,  $y = \ln(x)$ , is defined for all positive  $x$ . The base of this function,  $e$ , is called Euler's number, and can be evaluated

in a variety of ways. The most common are:  $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$  or  $e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ .

The exponential laws correspond to most of the laws of logarithms given above, namely:

$$a^0 = 1, a \neq 0$$

$$a^1 = a$$

$$a^x a^y = a^{x+y}$$

$$(a^x)^y = a^{xy}$$

$$a^{-1} = \frac{1}{a}$$

$$\frac{1}{a^x} = \left(\frac{1}{a}\right)^x = a^{-x}$$

$$a^x b^x = (ab)^x$$

$$\sqrt[n]{a^b} = \left(\sqrt[n]{a}\right)^b = a^{\frac{b}{n}}$$

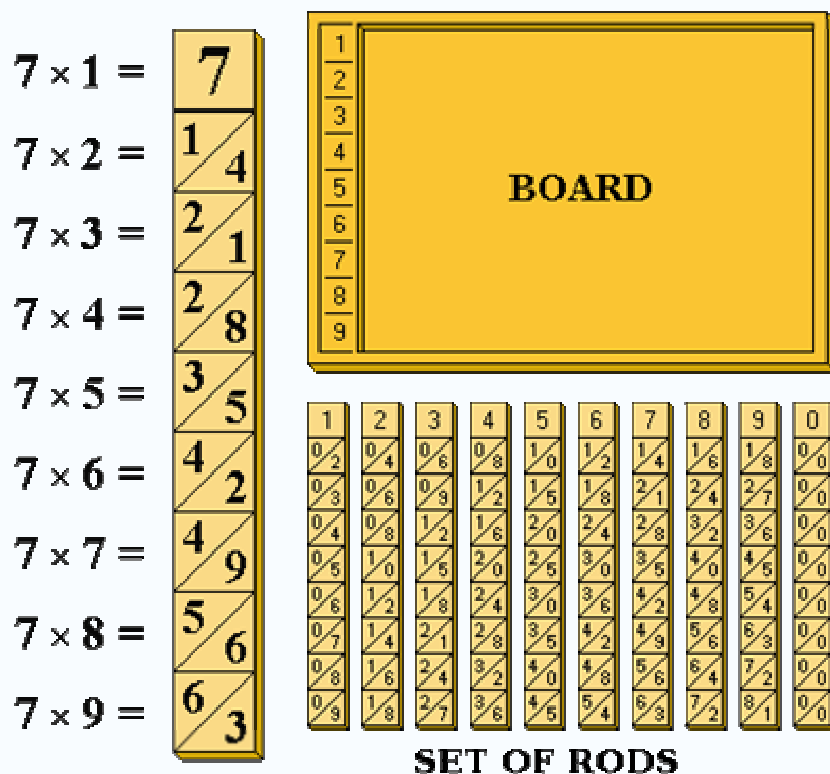
## Trigonometric Functions

Inverse trigonometric functions (such as  $\sin^{-1}x$  or  $\tan^{-1}x$ ) are functions that return the angle from its given trigonometric ratio. The inverse trigonometric functions are **multivalued**. For example, there are multiple values of  $x$  such that  $y = \sin x$ , so  $\sin^{-1}y$  is not uniquely defined unless a principal value is defined. Such principal values are sometimes denoted with a capital letter so, for example, the principal value of the inverse sine,  $\sin^{-1}x$ , may be variously denoted  $\text{Sin}^{-1}x$  or by  $\arcsin x$ . On the other hand, the notation  $\sin^{-1}x$  is also commonly used to denote either the principal value or *any* quantity whose sine is  $x$ . Worse still, the principal value and multiple valued notations are sometimes reversed, with  $\sin^{-1}x$  denoting the principal value and  $\arcsin x$  denoting the multivalued functions.

Different conventions are possible for the domain and range of these functions for the purpose of keeping them as single-valued functions; the most in use are illustrated below.

Function name	Function	Domain	Range
inverse sine	$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$
inverse cosine	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
inverse tangent	$\tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$
inverse cosecant	$\csc^{-1} x$	$(-\infty, \infty)$	$\left[-\frac{1}{2}\pi, 0\right)$ or $\left(0, \frac{1}{2}\pi\right]$
inverse secant	$\sec^{-1} x$	$(-\infty, \infty)$	$\left[0, \frac{1}{2}\pi\right)$ or $\left(\frac{1}{2}\pi, \pi\right]$
inverse cotangent	$\cot^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{1}{2}\pi, 0\right)$ or $\left(0, \frac{1}{2}\pi\right]$

**Napier's bones** are an abacus invented by John Napier (born in Merchiston Tower, in 1550) Edinburgh, for calculation of products and quotients of numbers. Also called **Rabdology** (from Greek ραβδος [rabdos], rod and λόγος [logos], word). Napier published his invention of the rods in a work printed in Edinburgh, Scotland, at the end of 1617 entitled *Rabdologiae*. Using the multiplication tables embedded in the rods, multiplication can be reduced to addition operations and division to subtractions. More advanced use of the rods can even extract square roots. Note that **Napier's bones** are not the same as logarithms, with which Napier's name is also associated.



The abacus consists of a board with a rim; the user places Napier's rods in the rim to conduct multiplication or division. The board's left edge is divided into 9 squares, holding the numbers 1 to 9. The **Napier's rods** consist of strips of wood, metal or heavy cardboard. **Napier's bones** are three dimensional, square in cross section, with four different **rods** engraved on each one. A set of such **bones** might be enclosed in a convenient carrying case.

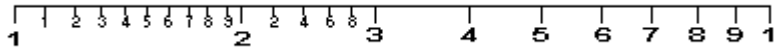
A rod's surface comprises 9 squares, and each square, except for the top one, comprises two halves divided by a diagonal line. The first square of each rod holds a single-digit, and the other squares hold this number's double, triple, quadruple and so on until the last square contains nine times the number in the top square. The digits of each product are written one to each side of the diagonal; numbers less than 10 occupy the lower triangle, with a zero in the top half.

A set consists of 9 rods corresponding to digits 1 to 9. The figure additionally shows the rod 0; although for obvious reasons it is not necessary for calculations.

# Slide Rules

In 1614, John Napier discovered the logarithm which made it possible to perform multiplications and divisions by addition and subtraction. (ie:  $a*b = 10^{(\log(a)+\log(b))}$  and  $a/b = 10^{(\log(a)-\log(b))}$ .)

This was a great time saver but there was still quite a lot of work required. The mathematician had to look up two logs, add them together and then look for the number whose log was the sum. Edmund Gunter soon reduced the effort by drawing a number line in which the positions of numbers were proportional to their logs.



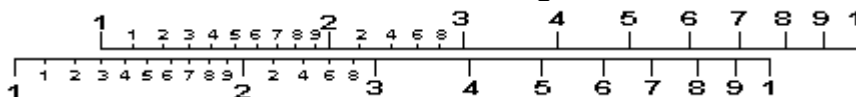
The scale started at one because the log of one is zero. Two numbers could be multiplied by measuring the distance from the beginning of the scale to one factor with a pair of dividers, then moving them to start at the other factor and reading the number at the combined distance.



The yellow spots are brass inserts to provide wear resistance at commonly used points.



Soon afterwards, William Oughtred simplified things further by taking two Gunter's lines and sliding them relative to each other thus eliminating the dividers.



In the years that followed, other people refined Oughtred's design into a sliding bar held in place between two other bars. Circular slide rules and cylindrical/spiral slide rules also appeared quickly. The cursor appeared on the earliest circular models but appeared much later on straight versions. By the late 17th century, the slide rule was a common instrument with many variations. It remained the tool of choice for many for the next three hundred years.

While great aids, slide rules were not particularly intuitive for beginners. A 1960 Pickett manual said:

"When people have difficulty in learning to use a slide rule, usually it is not because the instrument is difficult to use. The reason is likely to be that they don't understand the mathematics on which the instrument is based, or the formulas they are trying to evaluate.

Some slide rule manuals contain relatively exhaustive explanations of the theory underlying the operations. In this manual it is assumed that the *theory* of exponents, of logarithms, of trigonometry, and of the slide rule is known to the reader, or will be recalled or studied by reference to formal textbooks on these subjects."

A 1948 Stanley manual expressed a somewhat different opinion:

"The principles of logarithmic calculators are too well known to those likely to be interested for it to be necessary to enlarge upon the subject here, especially as it is absolutely unnecessary to have any knowledge of the subject to use the calculator"

...

"Anyone can calculate with the Fuller after a brief study of the following instructions **without any mathematical knowledge whatever.**"

Another interesting quote from the same Pickett manual:

"A computer who must make many difficult calculations usually has a slide rule close at hand."

In 1960, "computer" was still understood to be a *person* who computed. By contrast, a recent dictionary begins the only definition of "computer" with "An electronic machine..."

## Some Slide Rule Terms

### **Mannheim**

A standard single-face rule with scales to solve problems in multiplication, division, squares, square roots, reciprocals, trigonometry and logarithms.

### **Polyphase**

Like a Mannheim but added a scale for cubes and cube roots and an inverted C scale (CI) to make certain problems easier to solve. (Some manufacturers used Mannheim and Polyphase interchangeably.)

### **Phillips**

The single sided rule similar to a Polyphase but with an inverted A scale (typically labeled R) instead of an inverted C scale (CI).

### **Duplex**

A double-faced rule. Typically added three folded scales (CF, CIF, DF) to those of the Polyphase rule to make many problems easier to solve.

### **Trig**

A rule with scales for solving trigonometry problems (S, ST, T).

### **Decitrig**

A rule with the trigonometric scales (S, ST, T) marked in degrees and tenths of a degree.

### **Dual Base**

A rule with that read both common and natural logs.

### **Log Log**

A rule with scales for raising numbers to powers. (Scales usually started with LL)

### **Vector**

A rule with hyperbolic functions.

### **Combinations**

The above terms were often combined on more complex rules like Polyphase Duplex Decitrig. (In this case the double-sided duplex overrode the single sided assumption of Polyphase.)

**“Families of Powers”**

<b>Exponent</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>	<b>11</b>
<b>B A S E</b>	1	1	1	1	1	1	1	1	1	1	1	1
	1	2	4	8	16	32	64	128	256	512	1024	2048
	1	3	9	27	81	243	729	2187				
	1	4	16	64	256	1024	4096					
	1	5	25	125	625	3125						
	1	6	36	216	1296	7776						
	1	7	49	343	2401	16807						
	1	8	64	512	4096	32768						
	1	9	81	729	6561	59049						
	1	10	100	1000	10000	100000						

If  $y = \sin^{-1}x$  find cosy and tany.

If  $y = \sin^{-1}\left(\frac{\sqrt{5}}{3}\right)$  find cosy and tany.

Find the inverse of  $y = 4x^2 - 8$ . Restrict the domain of  $y = 4x^2 - 8$  so that its inverse is a function.

Sketch the graph of the inverse of the function  $y = f(x)$  illustrated in the diagram shown below.

