Factoring Polynomials

Some definitions (not necessarily all for secondary school mathematics):

A **polynomial** is the sum of one or more terms, in which each term consists of a product of a constant and one or more variables raised to some non-negative integer exponents.

A polynomial with only one term is called a **monomial**, with two terms is called a **binomial** and with three terms is called a **trinomial**.

A polynomial in variable x (a **single-variable polynomial**) is of the form:
$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$
where all of the coefficients (a_n, a_{n-1}, a_{n-2} , . . . + a_2 , a_1 , a_0) are real numbers and all of the exponents (n, $n-1$, $n-2$, . . . ,3 , 2, 1) are non-negative integers. The term a_0 is called the **constant term**.

The **degree** of a **term** of a polynomial is the sum of the exponents of the variables of that term. The **degree** of a **polynomial** is the **largest** degree of any of the individual terms of that polynomial.

The **root** of a polynomial $p(x)$ is a number, a, such that $p(a) = 0$. The **root** or ***x*-intercept** or **zero** of the corresponding polynomial equation $p(x) = 0$ is the number, a, that makes $p(a) = 0$. If a is a root of polynomial $p(x)$, then $p(x) = (x - a)(q(x)$ for some polynomial $q(x)$. The process of identifying some or all such factors $(x - a)$ is called **factoring** the polynomial. There are many different types of factoring techniques that can be applied, none of which work for all polynomials, so determining which factoring method to be applied requires some skill and persistence.

The **Fundamental Theorem of Algebra** states: Every polynomial equation having complex coefficients and degree $n \geq 1$ has at least one complex root. (On a lighter note, the **Frivolous Theorem of Arithmetic** states: Almost all natural numbers are very, very, very large.)

This can be written symbolically as:
$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 \\ &= a_n(x - z_n)(x - z_{n-1})(x - z_{n-2}) \ldots (x - z_2)(x - z_1) \\ &= a_n \prod_{i=1}^{n} (x - z_i) \qquad \text{(called \textbf{standard form})} \end{aligned}$$

The theorem can also be taken to mean that any polynomial with real coefficients has at least one root, which can be real or complex. It can be extended to imply that any polynomial with real coefficients can be can be factored into the product of linear or quadratic roots (which is applicable to secondary school mathematics).

For all remaining definitions and examples, we shall assume that all coefficients of the polynomials and the polynomial equations (or functions) are real numbers, unless otherwise stated.

A trinomial which has a leading coefficient of 1 is called a **monic** quadratic trinomial (e.g. $x^2 + 7x + 2$) and those which have the leading coefficient (i.e. “a” in a $ax^2 + bx + c$) not equal to 1 is called a **nonmonic** quadratic trinomial. This distinction usually requires a different approach when factoring. The term of the polynomial (in one variable) with the highest exponent is called the **leading term** or the **dominant term**. The polynomial also derives is name (in terms of its degree) from this leading term. Polynomials are classified as **linear** (for $ax + b$), **quadratic** ($ax^2 + bx + c$), **cubic** ($ax^3 + bx^2 + cx + d$), quartic ($ax^4 + bx^3 + cx^2 + dx + e$), etc., for leading term exponents of 1, 2, 3, 4, etc.

A polynomial of the form $(x - a)^n$ is said be **degenerate** (or to have degenerate roots) as all roots are equal to a.

A polynomial of degree n has **exactly** n roots. These roots are not all necessarily distinct. If one root r, of a polynomial occurs just once, then it is called a **simple root**. If root r occurs exactly m times, where $m > 1$, then it is called **a root of multiplicity *m*** (or an ***m*-fold root**). If $m = 2$, r is called a double root, if $m = 3$, it is called a triple root; and so on.

If a polynomial with **rational** coefficients has the **irrational root** $a + \sqrt{b}$ (where a and b are **rational** numbers) then its irrational conjugate, $a - \sqrt{b}$, is also a root.

If a polynomial with **real** coefficients has the **imaginary root** $a + \sqrt{b}$ (where a and b are **real** numbers) then its imaginary conjugate, $a - \sqrt{b}$, is also a root.

A polynomial has 0 as a root if and only if the constant term of the equation is zero.
If the leading coefficient of a polynomial is 1, then **all of its rational roots are integers**.

If a rational fraction $\frac{p}{q}$, expressed in lowest terms, is a root of a polynomial (or polynomial equation) then p is a divisor of the constant term a_0 and q is a divisor of the leading coefficient, a_n. Notice that the converse of this statement is **not** true. For example, we can only say that $\frac{2}{3}$ is a *possible* root of the polynomial $9x^4 - 5x^2 + 8x + 4$ (since 2 is a factor of 4 and 3 is a factor of 9). Unfortunately, this can necessitate a lengthy process of examining many potential roots of a polynomial before one or more is actually identified. Once we have found one root of a polynomial, it usually becomes easier to identify its remaining roots.

As well, there are a number of useful hints and strategies that can speed up this process (but you will have to attend the session on Saturday to learn these). One such rule (that will not be dealt with on Saturday) is briefly described here. It is called **Descartes' Rule of Signs for Polynomial Roots**. It first requires the definition of the variation of sign of a polynomial. When a polynomial is arranged in descending order of powers of the variable, if two successive terms differ in sign, the polynomial is said to have a **variation of sign**. For example, the polynomial $4x^5 - 2x^3 + 5x^2 + 12x - 4$ has three variations of sign. Descartes' Rule states that the number of positive roots of a polynomial is equal *either* to the number of variations of sign or to that number diminished by an even number. (It is not all that helpful for factoring polynomials at the high school level is it!)

Each attempt made to confirm that a possible root $(x - a)$ for polynomial $p(x)$ really is a root (using either long division or synthetic division) is called a **trial**. If it turns out that $(x - a)$ is a root of $p(x)$ then it is called a **successful trial**, and it is called a **false trial** if it is unsuccessful. Once you have identified one root of the polynomial equation $p(x) = 0$, and have used long or synthetic division to find $q(x)$, the remaining factor of $p(x)$ (i.e. $p(x) = (x - a)q(x) = 0$), then the resulting equation, $q(x) = 0$ is called the **depressed equation** for $p(x) = 0$.

The first few expansions of the binomial theorem $(x + y)^n$ are often required in working with polynomials, and also lead to the definitions of certain types of factoring questions. These are:

$(x + y)^2 = x^2 + 2xy + y^2$ and $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ and $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

These lead to the definitions or formulas for factoring perfect square trinomials and can also extend to factoring the difference of squares and the sum and difference of cubes. We have:

$x^2 - y^2 = (x - y)(x + y)$ and $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ and $x^3 + y^3 = (x - y)(x^2 - xy + y^2)$

Some of the different methods used to factor polynomials likely include:

Grade	**9**	**10**	**11**	**12**
T	Common factor	Common factor	Common factor	Common factor
e		Difference of squares	Difference of squares	Difference of squares
c		Perfect square trinomials	Perfect square trinomials	Perfect square trinomials
h		Simple trinomials	Grouping	Sum of cubes
n		Complex trinomials	Simple trinomials	Difference of cubes
i			Complex trinomials	Grouping
q				Simple trinomials
u				Complex trinomials
e				The Factor Theorem

The basic skills in mathematics needed to apply the techniques include a strong working knowledge of:

The multiplication **and division** facts up to 12 × 12 (or to 15 × 15 or even to 20 × 20???)
Perfect squares up to 12^2 (or up to 15^2 or even up to 30^2)
Perfect cubes up to 10^3

Here is a drill sheet that can be used for practicing the rapid factoring of monic quadratic trinomials:

1.)	$(x \quad)(x \quad)$	2.)	$(x \quad)(x \quad)$
3.)	$(x \quad)(x \quad)$	4.)	$(x \quad)(x \quad)$
5.)	$(x \quad)(x \quad)$	6.)	$(x \quad)(x \quad)$
7.)	$(x \quad)(x \quad)$	8.)	$(x \quad)(x \quad)$
9.)	$(x \quad)(x \quad)$	10.)	$(x \quad)(x \quad)$

You can work out the following questions ahead of the session if you like but it is certainly not necessary to do so. Note that there is not enough space on this page to fully factor the longer types.

1.) Factor the following trinomials: (designated as "simple" above)

a) $x^2 + 11x + 24$

b) $x^2 - 11x + 24$

c) $x^2 + 11x - 24$

d) $x^2 - 11x - 24$

Sign	**Pattern**
Question	**Answer**
+ +	+ +
– +	– –
+ –	} one + and
– –	} one – sign
(largest number has	sign of the x-term)

2.) Factor the following trinomials: (designated as "complex" above)

a) $5x^2 + 16x + 12$

b) $4x^2 - 19x + 21$

c) $6x^2 + 13x - 5$

d) $16x^2 - 40x + 25$

e) $16x^2 - 58x + 25$

3.) Factor the following trinomials: (common factoring required)

a) $15x^2y^5 + 25x^4y^3 + 5x^2y^2$

b) $4x(7m - 5) - 8y(5 - 7m)$

c) $2\pi r^2 + 4\pi rh$

d) $e^{x^2-x} + (2x-1)e^{x^2-x}$

e) $\frac{5x}{4y^2} - \frac{7}{4y} + \frac{9z}{2y}$

f) $6(x-4)^{\frac{1}{3}}(x^2+3x)^5 + \frac{1}{3}(x-4)^{-\frac{2}{3}}(x^2+3x)^6$

4.) Factor the following trinomials: (difference of squares or cubes)

a) $225x^2 - 169y^2$

b) $\Omega^2 - (3v - 2)^2$

c) $4x^2 - 20x + 25 - 9x^2 + 42x - 49$

d) $x^6 - y^6$

e) $125x^3 + 343$

(Use difference of squares) Evaluate: 47×53

5.) Divide: $3x^4 - 5x^2 + 6x - 4$ by $x - 2$

using a) formal long division *or*

b) synthetic division (Horner's Method for evaluating a function in x, given $x = a$)

	3	−5	6	−4
2				

6.) Factor the following trinomials: (factor theorem may be required)

a) $x^3 - 3x^2 - 10x + 24$

b) $x^3 + 9x^2 + 23x + 15$

c) $2x^3 + 10x^2 - 8x - 40$

d) $16x^2 - 40x - 25$

7.) Solve for x: a) $x^2 = x + 1$

b) $8x^4 - 8x^2 - x + 1 = 0$

(Do you recognize the significance of these two equations?)

Note that $\phi = \dfrac{1+\sqrt{5}}{2}$ (See the attached Excel sheet on the Golden Mean.)

The Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The general formula for solving the cubic equation $ax^3 + bx^2 + cx + d = 0$ **is given by:**

$$x = \frac{1}{3a}\Big\{ -b + \left[-2b^3 + 9abc - 27d^2 + 3\sqrt{-3a^2b^2c^2 + 81a^4d^2 - 54a^3bcd + 12a^2b^3d + 12a^3c^3}\right]^{\frac{1}{3}} \cdot 2^{\frac{1}{3}}$$
$$+ \left[-2b^3 + 9abc - 27d^2 - 3\sqrt{-3a^2b^2c^2 + 81a^4d^2 - 54a^3bcd + 12a^2b^3d + 12a^3c^3}\right]^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \Big\}$$

A Closer Look at The Difference of Squares

Specific Expectations

Manipulating Quadratic Expressions

By the end of this course, students will:

- expand and simplify second-degree polynomial expressions involving one variable that consist of the product of two binomials [e.g., $(2x + 3)(x + 4)$] or the square of a binomial [e.g., $(x + 3)^2$], using a variety of tools (e.g., algebra tiles, diagrams, computer algebra systems, paper and pencil) and strategies (e.g. patterning);
- factor binomials (e.g., $4x^2 + 8x$) and trinomials (e.g., $3x^2 + 9x - 15$) involving one variable up to degree two, by determining a common factor using a variety of tools (e.g., algebra tiles, computer algebra systems, paper and pencil) and strategies (e.g., patterning);
- factor simple trinomials of the form $x^2 + bx + c$ (e.g., $x^2 + 7x + 10$, $x^2 + 2x - 8$), using a variety of tools (e.g., algebra tiles, computer algebra systems, paper and pencil) and strategies (e.g., patterning);
- factor the difference of squares of the form $x^2 - a^2$ (e.g., $x^2 - 16$).

Specific Expectations

Solving Quadratic Equations

By the end of this course, students will:

- pose and solve problems involving quadratic relations arising from real-world applications and represented by tables of values and graphs (e.g., "From the graph of the height of a ball versus time, can you tell me how high the ball was thrown and the time when it hit the ground?");
- represent situations (e.g., the area of a picture frame of variable width) using quadratic expressions in one variable, and expand and simplify quadratic expressions in one variable [e.g., $2x(x + 4) - (x + 3)^2$];*
- factor quadratic expressions in one variable, including those for which a ≠ 1 (e.g., $3x^2 + 13x - 10$), differences of squares (e.g., $4x^2 - 25$), and perfect square trinomials (e.g., $9x^2 + 24x + 16$), by selecting and applying an appropriate strategy (*Sample problem:* Factor $2x^2 - 12x + 10$.);*
- solve quadratic equations by selecting and applying a factoring strategy;

- factor quadratic expressions in one variable, including those for which a ≠ 1 (e.g., $3x^2 + 13x - 10$), differences of squares (e.g., $4x^2 - 25$), and perfect square trinomials (e.g., $9x^2 + 24x + 16$), by selecting and applying an appropriate strategy (*Sample problem:* Factor $2x^2 - 12x + 10$.);*
- solve quadratic equations by selecting and applying a factoring strategy;

Manipulating Algebraic Expressions

By the end of this course, students will:

- demonstrate an understanding of the remainder theorem and the factor theorem;
- factor polynomial expressions of degree greater than two, using the factor theorem;
- determine, by factoring, the real or complex roots of polynomial equations of degree greater than two;

The following are in increasing order of difficulty or complexity (as you go down, then across)

$x^2 - 16$	$x^4 - 25$	$(x - 5)^2 - (2y + 3)^2$	$4x^2 + 12x + 9 - y^2 - 6y - 9$
$9x^2 - 16$	$x^6 - 25y^{10}$	$9(x - 5)^2 - 16(2y + 3)^2$	$16x^2 + 40x + 25 - 9x^2 - 48y - 64$
$9x^2 + 16$	$x^4 - 81$	$x^2 + 10x + 25 - 49y^2$	$4x^2 + 6z + 25y^2 - z^2 - 9 + 20xy$
$9x^2 - 16y^2$	$(x + 5)^2 - 36$	$9y^2 - x^2 + 6x - 9$	$x^4 + 4x^2 + 16$
$24x^2 - 54$	$36 - (x - 5)^2$		

Some Selected Number Squaring Patterns

Squaring Numbers Ending In 5	**Squaring Numbers in the 50's**	**Squaring Numbers in the 100's**	**Squaring Numbers in the 90's**
15^2 = 225	50^2 = 2500	101^2 = 10201	99^2 = 9801
25^2 = 625	51^2 = 2601	102^2 = 10404	98^2 = 9602
35^2 = 1225	52^2 = 2704	103^2 = 10609	97^2 = 9409
45^2 =	53^2 = 2809	104^2 =	96^2 = 9216
55^2 =	54^2 =	105^2 =	95^2 =
65^2 =	55^2 =	106^2 =	94^2 =
75^2 =	56^2 =	107^2 =	93^2 =
85^2 =	57^2 =	108^2 =	92^2 =
95^2 =	58^2 =	109^2 =	91^2 =
995^2 =	59^2 =	114^2 =	90^2 =

Why the First Two Patterns Work

1.) e.g. $57^2 = 2500 + 700 + 49$

Let the general term be $50 + n$, $n \in I^+$

$$\begin{aligned} \text{Now } (50 + n)^2 &= 50(50 + n) + n(50 + n) \\ &= 2500 + 50n + 50n + n^2 \\ &= 2500 + 100n + n^2 \\ &= 100(25 + n) + n^2 \end{aligned}$$

2.) e.g. $65^2 = 100 \times (6 \times 7) + 25$

Let the general number be $10n + 5$, $n \in I^+$

$$\begin{aligned} \text{Now } (10n + 5)^2 &= 10n(10n + 5) + 5(10n + 5) \\ &= 100n^2 + 50n + 50n + 25 \\ &= 100n^2 + 100n + 25 \\ &= 100(n^2 + n) + 25 \\ &= 100(n)(n+1) + 25 \end{aligned}$$

Divisibility Tests

Dividing by 2

Look at the ones digit. If it is divisible by 2, then the number is as well.
Example:

327829338926486482642183756 is divisible by four also, because 6 is divisible by two.

Dividing by 3

Add up the digits: if the sum is divisible by three, then the number is as well.
Examples:

111111: the digits add to 6 so the whole number is divisible by three.

87687687. The digits add up to 57, and 5 +7 = 12, so the original number is divisible by three.

Dividing by 4

Look at the last two digits (the tens and ones digits). If they are divisible by 4, the number is as well.
Example:

1. 1732782989264864826421834612 is divisible by four also, because 12 is divisible by four.

Dividing by 5

If the ones digit is a five or a zero, then the number is divisible by 5.

Dividing by 6

Check the rule for divisibility by 2 and by 3. If the number is divisible by both 2 and 3, it is divisible by 6 as well.

Dividing by 7

To find out if a number is divisible by seven, take the ones digit, double it, and subtract it from the rest of the number.
Example: If you had 203, you would double the ones digit to get six, and subtract that from 20 to get 14. If you get an answer divisible by 7 (including zero), then the original number is divisible by seven. If you don't know the new number's divisibility, you can apply the rule again.

Here is another method for divisibility by seven: Divide off the number into groups of 3 digits (from the right). You may be left with a group of one or two digits on the left end. Then find the sum of the alternate groups of these new 3-digit numbers. If the difference between the two sums is divisible by 7, then so was the original number.

Example: For the number 1425865742665 divide up the number into these groups of digits 1 425 865 742 665. Now add up the alternate groups:
1 + 865 + 665 = 1531 and 425 + 742 = 1167
Find the difference between theses new sums: 1531 – 1167 = 364
Since 364 is evenly divisible by 7 (52 times) then 1425865742665 also was divisible by 7.

Dividing by 8

Check the last three digits (the hundreds, tens and ones digits). If the last three digits of a number are divisible by 8, then so is the whole number.
Example: **33333888** is divisible by 8; 33333886 isn't.

Dividing by 9

Add the digits. If they are divisible by nine, then the number is as well. This holds for any power of three.

Dividing by 10

If the units digit is a 0, then the number is divisible by 10.

Dividing by 11

Take any number, such as **365167484**.

Add the first, third, fifth, seventh,.., digits.....3 + 5 + 6 + 4 + 4 = **22**

Add the second, fourth, sixth, eighth,.., digits.....6 + 1 + 7 + 8 = **22**

If the difference, including 0, is divisible by 11, then so is the number.

22 - 22 = 0 so **365167484** is evenly divisible by **11**.

Dividing by 12

Check for divisibility by 3 and 4.

Dividing by 13

Delete the ones digit from the given number. Then subtract nine times the deleted digit from the remaining number. If what is left is divisible by 13, then so is the original number.

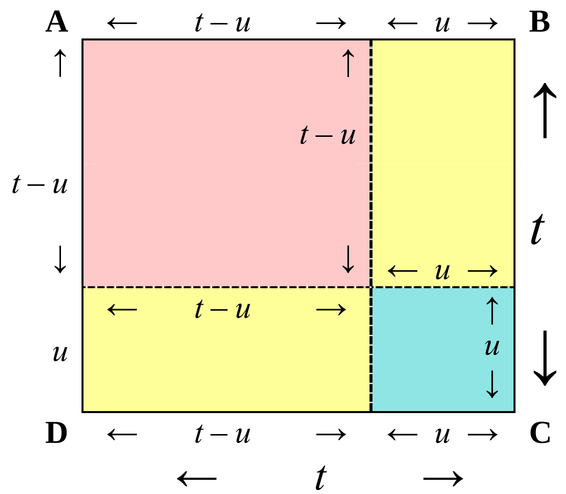
Example: For the number 4667, delete last 7 and multiply it by 9 (7 × 9 = 63).
Now 466 – 63 = 403. Since 403 is divisible by 13 (31 times) so too is 4667.

Another method for divisibility tests for 11 and for 13 involves exactly the same process as the second method given for divisibility by 7. Divide the number into groups of three and find the sum of alternating groups. If the difference of these sums is divisible by 11 (or by 13) then the original number was also divisible by 11 (or 13).

An Alternate Proof for The Quadratic Formula

Let $x = t - u$ (where $t > u$)

Consider square ABCD with side of length t (shown at right)



Let u be the length of the side of a smaller square (shaded in blue in the diagram at right.

Thus the pink square (shown at right) has sides of length $t - u$. We also have two rectangles (shown in yellow), each with length $t - u$ and width u.

The area of large square (with side of length t) equals the sum of the areas of the two smaller (in pink and blue) squares and the two (yellow) rectangles. Thus we have:

$$t^2 = (t-u)^2 + 2u(t-u) + u^2$$

which gives: $$t^2 - u^2 = (t-u)^2 + 2u(t-u)$$

Now let $m = 2u$ and $n = t^2 - u^2$

Hence, $u = \frac{m}{2}$ and thus $t^2 - u^2 = t^2 - \frac{m^2}{4} = n$ Therefore $t^2 = \frac{m^2}{4} + n$ which gives $t = \pm\sqrt{\frac{m^2}{4} + n}$

Therefore $x = t - u = \pm\sqrt{\frac{m^2}{4} + n} - \frac{m}{2} = -\frac{m}{2} \pm \sqrt{m^2 + 4n} = \frac{-m \pm \sqrt{m^2 + 4n}}{2}$

Since $t^2 - u^2 = (t-u)^2 + 2u(t-u)$ we get $(t-u)^2 = t^2 - u^2 - 2u(t-u)$

or $x^2 = n - mx$ which gives us $x^2 + mx - n = 0$ and the solution for x in this equation is:

$$\boxed{x = \frac{-m \pm \sqrt{m^2 + 4n}}{2}}$$

Of course, $x^2 + mx - n = 0$ can be expressed as $ax^2 + bx + c = 0$ if we let $m = \frac{b}{a}$ and $n = -\frac{c}{a}$

This would make the solution for $ax^2 + bx + c = 0$ result in $x = \frac{-\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}}{2}$ which gives

$$x = \frac{-\frac{b}{a} \pm \sqrt{\frac{b^2 - 4ac}{a^2}}}{2} = \frac{-\frac{b}{a} \pm \frac{\sqrt{b^2 - 4ac}}{a}}{2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hints to use to identify possible roots when using the Factor Theorem
(In the examples below, we mean that if ($x - a$) is a factor then "a" is a root)

1.) For the polynomial $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_1 x + a_0$ (or the corresponding polynomial equation):

a) If p is a divisor of the constant term a_0 then $a \pm p$ is a possible divisor of the polynomial, and

b) If q is a divisor of the leading coefficient, a_n then $qx \pm p$ is a possible factor of the polynomial

For example, the polynomial $3x^3 + 6x^2 - 8x - 10$ *could have* roots ±1, ±2, ±5, ±10 $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}$ and $\pm\frac{10}{3}$. (You must still test one or more of these roots until you obtain a successful trial.)

2.) If the signs alternate from term to term in the polynomial (**including** the signs of terms with coefficient zero), then at least one root is positive. Examples of this type of polynomial include: $5x^3 - 9x^2 + 7x - 5$ and $2x^4 + 6x^2 - 8x + 5$ (note that it must be written: $2x^4 - 0x^3 + 6x^2 - 8x + 5$)

3.) If the signs of **every** term of the polynomial are positive, then **all** of the rational roots of the polynomial are negative. e.g. all of the rational roots of $x^4 + 5x^3 + 7x^2 + 8x + 4$ are negative

4.) If the absolute value of the largest coefficient of any term of the polynomial is larger than the sum of the absolute values of each of the remaining terms, then **neither** 1 **nor** −1 is a root of that polynomial. e.g. in the polynomial $2x^3 + 16x^2 - 5x - 4$, neither 1 nor −1 can be a root (because 16 > 2 + 5 + 4)

5.) If the sum of the coefficients of the terms of a polynomial are zero, then 1 is a root of that polynomial. e.g. 1 is a root of $x^4 - 9x^3 + 4x^2 + 8x - 4$ because $1 - 9 + 4 + 8 - 4 = 0$

6.) This one is a little more difficult to see at first glance, and it really only works because the text book authors are making up questions that can be factored. If you are able to make the sum of the coefficients of all terms of the polynomial zero by changing one or more signs, then −1 is a root of the polynomial. Examples of this type are: $x^4 + 5x^3 - 7x^2 + 9x + 4$ (because $1 + 5 + 7 - 9 - 4$ equals 0) and $2x^4 - 3x^3 + 8x^2 + 2x + 5$ (because $2 + 3 - 8 - 2 + 5 = 0$).

7.) **Descartes' Rule of Signs for Polynomial Roots**. This first requires the definition of the variation of sign of a polynomial. When a polynomial is arranged in descending order of powers of the variable, if two successive terms differ in sign, the polynomial is said to have a **variation of sign**. For example, the polynomial $4x^5 - 2x^3 + 5x^2 + 12x - 4$ has three variations of sign. Descartes' Rule states that the number of positive roots of a polynomial is equal *either* to the number of variations of sign or to that number diminished by an even number. Be aware that it is **not** necessary to insert missing terms with a zero coefficient (such as $0x^4$ in the example above) when using this rule, but it is necessary when using the rule in point (2.) given above.

The suggested method for testing potential roots in the quickest possible fashion is to use synthetic division. This method not only distinguishes between false and successful trials faster than can be obtained by using a calculator, but it also identifies the depressed factor (i.e. the other factor).